

# Log-concavity and symplectic flows

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## Abstract

Let  $(M, \omega)$  be a compact, connected symplectic  $2n$ -dimensional manifold on which an  $(n - 2)$ -dimensional torus  $T$  acts effectively and Hamiltonianly. Under the assumption that there is an effective complementary 2-torus acting on  $M$  with symplectic orbits, we show that the Duistermaat-Heckman measure of the  $T$ -action is log-concave. This verifies the logarithmic concavity conjecture for a class of inequivalent  $T$ -actions. Then we use this conjecture to prove the following: if there is an effective symplectic action of an  $(n - 2)$ -dimensional torus  $T$  on a compact, connected symplectic  $2n$ -dimensional manifold that admits an effective complementary symplectic action of a 2-torus with symplectic orbits, then the existence of  $T$ -fixed points implies that the  $T$ -action is Hamiltonian. As a consequence of this, we give new proofs of a classical theorem by McDuff about  $S^1$ -actions, and some of its recent extensions.

## 1 Introduction

Let  $T$  be a torus, i.e. a compact, connected, commutative Lie group. This paper proves the logarithmic concavity conjecture of the Duistermaat-Heckman density function for a class of inequivalent Hamiltonian  $T$ -actions on compact, connected, symplectic manifolds  $(M, \omega)$  of dimension  $4 + 2 \dim T$ . In particular, we address the case of  $S^1$ -actions on 6-dimensional manifolds, which has received much interest recently.

Then we use logarithmic concavity methods to show that the Hamiltonian character of the symplectic  $T$ -action is equivalent to the fact that there are some  $T$ -fixed points. Our results imply, with different proofs, a classical theorem of McDuff about circle (i.e. 1-dimensional tori) actions on 4-manifolds [28] (see Corollary 5.2) and its extension to complexity-1 actions by Kim [20] (see Theorem 5.1 and Section 6).

In fact, one of the fundamental questions in equivariant symplectic geometry is to what extent there is a close relationship between: (i) a symplectic  $T$ -action being Hamiltonian; and (ii) the symplectic  $T$ -action having fixed points. In the late fifties Frankel [8] proved that on compact connected Kähler manifolds, a  $2\pi$ -periodic vector field whose flow preserves the Kähler form (equivalently, a Kähler  $S^1$ -action) is Hamiltonian if and only if has a fixed point. This easily implies that in the compact, connected, Kähler setting, conditions (i) and (ii) are equivalent for any torus  $T$  (see for instance Step 4 in [34, Proof of Theorem 3]).

In the early eighties, McDuff proved that if the fixed point set of a symplectic action of a circle on a compact connected symplectic four manifold is non-empty, then the action must be Hamiltonian. However, she also constructed an example of a symplectic circle on a compact, connected symplectic six manifold with non-empty fixed point set which is not Hamiltonian. So while it is true that if a symplectic  $T$ -action on a compact, connected symplectic manifold is Hamiltonian, then it does have fixed points, the converse does not hold in general in dimension six. However, McDuff's result did not have a transversal toric symmetric, in which case symplectic flows be Hamiltonian. Precisely, we have the following.

**Theorem 1.1.** *Let  $(M, \omega)$  be a compact, connected, symplectic 6-dimensional manifold equipped with an effective symplectic action of a 2-torus  $S$  on  $M$  whose orbits are symplectic. Let  $\{\varphi_t\}_{t \in \mathbb{R}}$  be a  $2\pi$ -periodic effective symplectic flow which commutes with the flows generated by the  $S$ -action. If  $\{\varphi_t\}_{t \in \mathbb{R}}$  has equilibrium points, then it is a Hamiltonian flow.*

**Remark 1.2.** Theorem 1.1 can be stated as follows. Let the circle  $S^1$  act effectively and with non-empty fixed point set on a compact, connected, symplectic 6-dimensional manifold  $(M, \omega)$ . Suppose that there is an effective commuting symplectic action of a 2-torus  $S$  on  $M$  whose orbits are symplectic. Then the action of  $S^1$  on  $(M, \omega)$  is Hamiltonian.

Theorem 1.1 is a particular case of the more general Theorem 1.3 below.

The problem of relating (i) and (ii) is still open in most cases, see Section 6. In the present paper we study this problem by studying another problem which is very interesting on its own right, and has generated a lot of activity: the log-concavity conjecture for Duistermaat-Heckman density functions.

In what follows we assume that  $n$  is an integer with  $n \geq 2$  (though the case  $n = 2$  is almost always trivial).

## Symplectic actions with transversal symplectic symmetry

Consider the symplectic action of a torus  $T$  on a compact, connected symplectic manifold  $(M, \omega)$  of dimension  $2n$ . The integer  $n - \dim T$  is called the *complexity of the  $T$ -action*. In view of McDuff's example, if the complexity of a symplectic  $T$ -action on  $M$  is two, the action does not have to be Hamiltonian even if the fixed point set is non-empty.

Nevertheless, the following theorem asserts that if the action of  $T$  has complexity two, and if there is a transversal toral symmetry with symplectic orbits, then the action of  $T$  must be Hamiltonian provided the fixed point set of the action is non-empty.

**Theorem 1.3.** *Let  $T$  be an  $(n - 2)$ -dimensional torus which acts effectively with non-empty fixed point set on a compact, connected, symplectic  $2n$ -dimensional manifold  $(M, \omega)$ . Suppose that there is an effective commuting symplectic action of a 2-torus  $S$  on  $M$  whose orbits are symplectic.<sup>1</sup> Then the action of  $T$  on  $(M, \omega)$  is Hamiltonian.*

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<sup>1</sup>the following statements are equivalent ([33, Corollary 2.2.4]): let  $(M, \omega)$  be a compact connected

We will prove Theorem 1.3 using logarithmic concavity techniques and the structure theory of symplectic actions. The proof combines several tools developed by others, and which we adapt to the setting of the paper: Lerman's symplectic cutting method [21], Yan's primitive decomposition of differential forms [44], Graham's work on logarithmic concavity of push-forward measures [11], the Lerman-Guillemin-Sternberg jump formulas [12], and Hodge-Riemann bilinear relations.

## Logarithmic concavity of Duistermaat-Heckman densities

Our second theorem is a proof of the log-concavity conjecture of Duistermaat-Heckman functions of Hamiltonian torus actions with transversal toral symmetry with symplectic orbits. We will prove this prior to proving Theorem 1.3 because its proof uses this theorem. In order to state it, consider the momentum map  $\mu: M \rightarrow \mathfrak{t}^*$  of the Hamiltonian  $T$ -action in Theorem 1.3, where  $\mathfrak{t}$  is the Lie algebra of  $T$ , and  $\mathfrak{t}^*$  its dual Lie algebra.

If  $V \subset M$  is an open set, the *Liouville measure on  $M$*  of  $V$  is defined as

$$\int_V \frac{\omega^n}{n!}.$$

The *Duistermaat-Heckman measure* on  $\mathfrak{t}^*$  is the push-forward of the Liouville measure on  $M$  by  $\mu$ . The Duistermaat-Heckman measure is absolutely continuous with respect to the Lebesgue measure, and its density function<sup>2</sup>

$$\mathrm{DH}_T: \mathfrak{t}^* \rightarrow [0, \infty)$$

is the *Duistermaat-Heckman function* (see Section 2). Recall that if  $V$  is a vector space, and if a Borel measurable function  $f: V \rightarrow \mathbb{R}$  is positive almost everywhere, we say that  $f$  is *log-concave* if its logarithm is a concave function.

**Theorem 1.4.** *Let  $T$  be an  $(n - 2)$ -dimensional torus which acts effectively and Hamiltonianly on a compact, connected, symplectic  $2n$ -dimensional manifold  $(M, \omega)$ . Suppose that there is an effective commuting symplectic action of a 2-torus  $S$  on  $M$  whose orbits are symplectic. Then the Duistermaat-Heckman function  $\mathrm{DH}_T$  of the Hamiltonian  $T$ -action is log-concave, i.e. its logarithm is a concave function.*

The log-concavity of the Duistermaat-Heckman function was proved by Okounkov [30, 31] for projective algebraic varieties, and by Graham [11] for arbitrary compact Kähler manifolds. In view of these positive results, it was conjectured independently by Ginzburg and Knudsen in 1990's that the Duistermaat-Heckman function of any Hamiltonian torus action on a compact symplectic manifold is log-concave.

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symplectic manifold equipped with an effective symplectic action of a 2-torus  $S$ : a) *at least one* principal  $S$ -orbit is a symplectic submanifold; b) *every* principal  $S$ -orbit is a symplectic submanifold; c)

*at least one*  $S$ -orbit is a 2-dimensional symplectic submanifold; d) *every*  $S$ -orbit is a 2-dimensional symplectic submanifold; e) *every*  $S$ -orbit is a symplectic submanifold.

<sup>2</sup>which is well defined once the normalization of the Lebesgue measure is declared

However, later Karshon [18] gave an example of a Hamiltonian circle action on a compact connected symplectic 6-manifold with a non log-concave Duistermaat-Heckman function, hence disproving the conjecture in general. More recently, Lin found a general construction of six dimensional Hamiltonian circle manifolds with non-log-concave Duistermaat-Heckman functions, and established the log-concavity conjecture for a class of Hamiltonian torus actions with complexity two [22].

## Applications: $S^1$ -actions and complexity-1 actions

We obtain, as a consequence of Theorems 1.3 and 1.4, a theorem of Kim (Theorem 5.1) that says that if  $(M, \omega)$  is a  $2n$ -dimensional compact, connected, symplectic manifold, then every effective symplectic action of an  $(n - 1)$ -dimensional torus on  $(M, \omega)$  with non-empty fixed point set is Hamiltonian. If  $n = 2$ , this statement is due to McDuff [28] (see Corollary 5.2). Our proofs are conceptually different, and use different tools, from that of Kim and McDuff. Theorems 1.3 and 1.4 cover wider classes of manifolds, see Example 5.6.

**Structure of the paper.** The rest of the paper is devoted to preparing the ground for the proofs, proving these two theorems, and exploring their applications.

In Section 2 we review the basic elements of symplectic manifolds and torus actions that we need in the remaining of the paper.

We prove Theorem 1.4 in Section 3, and we prove Theorem 1.3 in Section 4 (the proofs of Theorems 1.3 and 1.4 rely mainly on: (1) the structure theory of non-Hamiltonian symplectic torus actions [33]; (2) the logarithmic convexity theorem for push-forward measures [11]).

In Section 5 we explain how our theorems can be used to derive proofs of some well-known results.

In Section 6 we discuss some related results to these theorems.

## 2 Duistermaat-Heckman theory

This section gives a fast review of the necessary background to read this paper.

### Symplectic group actions

Let  $(M, \omega)$  be a symplectic manifold, i.e. the pair consisting of a smooth manifold  $M$  and a symplectic form  $\omega$  on  $M$  (a non-degenerate closed 2-form on  $M$ ). Let  $T$  be a torus, i.e. a compact, connected, commutative Lie group with Lie algebra  $\mathfrak{t}$  ( $T$  is isomorphic, as a Lie group, to a finite product of circles  $S^1$ ).

Suppose that  $T$  acts on a symplectic manifold  $(M, \omega)$  symplectically (i.e., by diffeomorphisms which preserve the symplectic form). We denote by

$$(t, m) \mapsto t \cdot m$$

the action  $T \times M \rightarrow M$  of  $T$  on  $M$ .

Any element  $X \in \mathfrak{t}$  generates a vector field  $X_M$  on  $M$ , called the *infinitesimal generator*, given by

$$X_M(m) := \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot m,$$

where  $\exp: \mathfrak{t} \rightarrow T$  is the exponential map of Lie theory and  $m \in M$ . As usual, we write

$$\iota_{X_M} \omega := \omega(X_M, \cdot) \in \Omega^1(M)$$

for the contraction 1-form. The  $T$ -action on  $(M, \omega)$  is said to be *Hamiltonian* if there exists a smooth invariant map  $\mu: M \rightarrow \mathfrak{t}^*$ , called the *momentum map*, such that for all  $X \in \mathfrak{t}$  we have that

$$\iota_{X_M} \omega = d\langle \mu, X \rangle, \quad (1)$$

where  $\langle \cdot, \cdot \rangle: \mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{R}$  is the duality pairing.

**Remark 2.1.** Theorem 1.3 says that there exists a smooth map invariant map  $\mu: M \rightarrow \mathfrak{t}^*$  satisfying (1), where  $\mathfrak{t}^*$  is the dual Lie algebra of  $T^{n-2}$ .

**Remark 2.2.** The existence of  $\mu$  is equivalent to the exactness of the 1-forms  $\iota_{X_M} \omega$  for all  $X \in \mathfrak{t}$ . The obstruction to the action to being Hamiltonian lies in the first de Rham cohomology group  $H_{\text{dR}}^1(M)$  of  $M$ , i.e. if  $H_{\text{dR}}^1(M)$  is trivial, then any symplectic  $T$ -action on  $M$  is Hamiltonian.

We say that the  $T$ -action on  $M$  has *fixed points* if

$$M^T := \{m \in M \mid t \cdot m = m, \text{ for all } t \in T\} \neq \emptyset. \quad (2)$$

The  $T$ -action is *effective* if the intersection of all stabilizer subgroups

$$T_m := \{t \in T \mid t \cdot m = m\}, \quad m \in M,$$

is the trivial group. The  $T$ -action is *free* if  $T_m$  is the trivial group for all points  $m \in M$ . The  $T$ -action is *semifree* if it is free on  $M \setminus M^T$ , where  $M^T$  is given by (2). Note that if a  $T$ -action is semifree, then it either leaves every point in  $M$  fixed, or it is effective. Finally, the action of  $T$  is called *quasi-free* if the stabilizer subgroup of every point are connected.

**Example 2.3.** The simplest example of a Hamiltonian torus action is given by the standard symplectic sphere  $S^2$  with the rotational  $S^1$ -action. This action is also effective and semifree. It is easy to check that the momentum map for this action is the height function  $\mu(\theta, h) = h$ .

## Atiyah-Guillemin-Sternberg convexity

We will need the following classical result.

**Theorem 2.4** (Atiyah [2], Guillemin and Sternberg [13]). *If a torus  $T$  with Lie algebra  $\mathfrak{t}$  acts on a compact, connected  $2n$ -dimensional symplectic manifold  $(M, \omega)$  in a Hamiltonian fashion, then the image  $\mu(M)$  under the momentum map  $\mu: M \rightarrow \mathfrak{t}^*$  of the action is a convex polytope  $\Delta \subset \mathfrak{t}^*$ .*

## Reduced symplectic forms and Duistermaat-Heckman formula

Consider the Hamiltonian action of a torus  $T$  on a symplectic manifold  $(M, \omega)$ . Let  $a \in \mathfrak{t}^*$  be a regular value of the momentum map  $\mu : M \rightarrow \mathfrak{t}^*$  of the action. When the action of  $T$  on  $M$  is not quasi-free, the quotient

$$M_a := \mu^{-1}(a)/T$$

taken at a regular value of the momentum map is not a smooth manifold in general. However,  $M_a$  admits a smooth orbifold structure in the sense of Satake [39].

Even though smooth orbifolds are not necessarily smooth manifolds, they carry differential structures such as differential forms, fiber bundles, etc. In fact, the usual definition of symplectic structures extends to the orbifold case. In particular, the restriction of the symplectic form  $\omega$  to the fiber  $\mu^{-1}(a)$  descends to a symplectic form  $\omega_a$  on the quotient space  $M_a$ , cf. Weinstein [42]. We refer to [1] and [29] for a modern account of basic notions in orbifold theory, and to [33, Appendix] for the basic definitions in the context of symplectic geometry.

We are ready to state an equivariant version of the celebrated Duistermaat-Heckman theorem.

**Theorem 2.5** (Duistermaat-Heckman [6]). *Consider an effective Hamiltonian action of a  $k$ -dimensional torus  $T^k$  on a compact, connected,  $2n$ -dimensional symplectic manifold  $(M, \omega)$  with momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . Suppose that there is another symplectic action of 2-torus  $T^2$  on  $M$  that commutes with the action of  $T^k$ . Then the following hold:*

- (a) *at a regular value  $a \in \mathfrak{t}^*$  of  $\mu$ , the Duistermaat-Heckman function is given by*

$$\text{DH}_{T^k}(a) = \int_{M_a} \frac{\omega_a^{n-k}}{(n-k)!},$$

*where  $M_a := \mu^{-1}(a)/T^k$  is the symplectic quotient,  $\omega_a$  is the corresponding reduced symplectic form, and  $M_a$  has been given the orientation of  $\omega_a^{n-k}$ ;*

- (b) *if  $a, a_0 \in \mathfrak{t}$  lie in the same connected component of the set of regular values of the momentum map  $\mu$ , then there is a  $T^2$ -equivariant diffeomorphism  $F : M_a \rightarrow M_{a_0}$ , where  $M_{a_0} := \mu^{-1}(a_0)/T^k$  and  $M_a := \mu^{-1}(a)/T^k$ . Furthermore, using  $F$  to identify  $M_a$  with  $M_{a_0}$ , the reduced symplectic form on  $M_a$  may be identified with*

$$\omega_a = a_0 + \langle c, a - a_0 \rangle,$$

*where  $c \in \Omega^2(M, \mathfrak{t}^*)$  is a closed  $\mathfrak{t}^*$ -valued two-form representing the Chern class of the principal torus bundle  $\mu^{-1}(a_0) \rightarrow M_{a_0}$ .*

**Example 2.6.** Under the conditions of Theorem 1.1, Theorem 1.4 says that the Duistermaat-Heckman function

$$a \mapsto \frac{1}{2} \int_{M_a} \omega_a^2,$$

is log-concave, where  $M_a$  is the symplectic quotient  $\mu^{-1}(a)/S^1$  and  $\omega_a$  is the reduced symplectic form inducing the orientation on  $M_a$ .

## Primitive decomposition of differential forms

Here we assume that  $(X, \omega)$  is a  $2n$ -dimensional symplectic orbifold. Let  $\Omega^k(X)$  be the space of differential forms of degree  $k$  on  $X$ . Let  $\Omega(X)$  be the space corresponding to the collection of all  $\Omega^k(X)$ , for varying  $k$ 's.

We note that, in analogy with the case of symplectic manifolds, there are three natural operators  $\Omega(X)$  on given as follows:

$$\begin{aligned} L : \Omega^k(X) &\rightarrow \Omega^{k+2}(X), \quad \alpha \mapsto \omega_a \wedge \alpha, \\ \Lambda : \Omega^k(X) &\rightarrow \Omega^{k-2}(X), \quad \alpha \mapsto \iota_\pi \alpha, \\ H : \Omega^k(X) &\rightarrow \Omega^k(X), \quad \alpha \mapsto (n - k)\alpha, \end{aligned}$$

where  $\pi = \omega^{-1}$  is the Poisson bi-vector induced by the symplectic form  $\omega$ . These operators satisfy the following bracket relations.

$$[\Lambda, L] = H, \quad [H, \Lambda] = 2\Lambda, \quad [H, L] = -2L.$$

Therefore they define a representation of the Lie algebra  $\mathfrak{sl}(2)$  on  $\Omega(X)$ . A primitive differential form in  $\Omega(X)$  is by definition a highest weight vector in the  $\mathfrak{sl}_2$ -module  $\Omega(X)$ . Equivalently, for any integer  $0 \leq k \leq n$ , we say that a differential  $k$ -form  $\alpha$  on  $X$  is primitive if and only if

$$\omega^{n-k+1} \wedge \alpha = 0.$$

Although the  $\mathfrak{sl}_2$ -module  $\Omega(X)$  is infinite dimensional, there are only finitely many eigenvalues of  $H$ . There is a detailed study of  $\mathfrak{sl}_2$ -modules of this type in [44]. The following result is an immediate consequence of [44, Corollary 2.5].

**Lemma 2.7.** *A differential form  $\alpha \in \Omega^k(X)$  admits a unique primitive decomposition*

$$\alpha = \sum_{r \geq \max(k-n, 0)} \frac{L^r}{r!} \beta_{k-2r}, \tag{3}$$

where  $\beta_{k-2r}$  is a primitive form of degree  $k - 2r$ .

## 3 Proof of Theorem 1.4

The proof of Theorem 1.4 combines ingredients coming from several directions: reduction theory for Hamiltonian group actions; structure theory for symplectic group actions with symplectic orbits; complex Duistermaat-Heckman theorem; Lerman-Guillemin-Sternberg wall jump formulas; Hodge-Riemann bilinear relations. We will first explain the particular form in which we need to use these ingredients above, and then how to combine them to prove the theorem.



## Toric fibers and reduction ingredients

We start with the following observations.

**Lemma 3.1.** *Let a torus  $T$  with Lie algebra  $\mathfrak{t}$  act on a compact, connected, symplectic manifold  $(M, \omega)$  in a Hamiltonian fashion with momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . Suppose that the action of another torus  $S$  on  $M$  is symplectic and commutes with the action of  $T$ . Then for any  $a \in \mathfrak{t}^*$ , the action of  $S$  preserves the level set  $\mu^{-1}(a)$  of the momentum map.*

*Proof.* Let  $X$  and  $Y$  be two vectors in the Lie algebra of  $S$  and  $T$  respectively, and let  $X_M$  and  $Y_M$  be the vector fields on  $M$  induced by the infinitesimal action of  $X$  and  $Y$  respectively. Let  $\mathcal{L}_{X_M}\mu^Y$  denote the Lie derivative of

$$\mu^Y := \langle \mu, Y \rangle : M \rightarrow \mathbb{R}$$

with respect to  $X_M$ . Let  $[\cdot, \cdot]$  denote the Lie bracket on vector fields.

It suffices to show that  $\mathcal{L}_{X_M}\mu^Y = 0$ .

Since the action of  $T$  and  $S$  commute, we have that

$$[X_M, Y_M] = 0.$$

By the Cartan identities, we have

$$\begin{aligned} 0 &= \iota_{[X_M, Y_M]}\omega = \mathcal{L}_{X_M}\iota_{Y_M}\omega - \iota_{Y_M}\mathcal{L}_{X_M}\omega \\ &= \mathcal{L}_{X_M}\iota_{Y_M}\omega \quad (\text{because the action of } S \text{ is symplectic.}) \\ &= \mathcal{L}_{X_M}(d\mu^Y) = d\mathcal{L}_{X_M}\mu^Y. \end{aligned}$$

This proves that  $\mathcal{L}_{X_M}\mu^Y$  must be a constant. On the other hand, we have

$$\mathcal{L}_{X_M}\mu^Y = \iota_{X_M}d\mu^Y = \iota_{X_M}\iota_{Y_M}\omega = \omega(Y_M, X_M).$$

Since the action of  $T$  is Hamiltonian and the underlying manifold  $M$  is compact, there must exist a point  $m \in M$  such that  $Y_M$  vanishes at  $m$ . Thus

$$(\mathcal{L}_{X_M}\mu^Y)(m) = 0,$$

and therefore  $\mathcal{L}_{X_M}\mu^Y$  is identically equal to zero on  $M$ . □

**Remark 3.2.** It is clear from the above argument that if the action of  $T$  is symplectic with a generalized momentum map, cf. Definition 4.2, and if the fixed point set of the  $T$ -action is non-empty, then the assertion of Lemma 3.1 still holds.



**Lemma 3.3.** *Let an  $(n - 2)$ -dimensional torus  $T^{n-2}$  with Lie algebra  $\mathfrak{t}$  act effectively on a compact connected symplectic manifold  $(M, \omega)$  in a Hamiltonian fashion with momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . Suppose that there is an effective commuting  $T^2$ -action on  $(M, \omega)$  which has a symplectic orbit. Then for any regular value  $a \in \mathfrak{t}^*$ , the symplectic quotient*

$$M_a := \mu^{-1}(a)/T^{n-2}$$

*inherits an effective symplectic  $T^2$ -action with symplectic orbits.*

*Proof.* Let  $a$  be a regular value of the momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . Then the symplectic quotient  $M_a$  is a four dimensional symplectic orbifold. By assumption, the  $T^2$ -action on  $M$  has a symplectic orbit. Hence all the orbits of the  $T^2$ -action on  $M$  are symplectic [33, Corollary 2.2.4].

By Lemma 3.1, the  $T^2$ -action preserves the level set  $\mu^{-1}(a)$ , and thus it descends to an action on the reduced space  $M_a$ . It follows that the orbits of the induced  $T^2$ -action on  $M_a$  are symplectic two dimensional orbifolds. Thus, by dimensional considerations, any isotropy subgroups of the  $T^2$ -action on  $M_a$  is zero-dimensional. Since these isotropy subgroups must be closed subgroups of  $T^2$ , they must be finite subgroups of  $T^2$ .

Now let  $H$  be the intersection of the all isotropy subgroups of  $T^2$ . Then  $H$  is a finite subgroup of  $T^2$  itself. Note that the quotient group  $T^2/H$  is a two-dimensional compact, connected commutative Lie group. Hence  $T^2/H$  is isomorphic to  $T^2$ . Therefore  $M_a$  admits an effective action of  $T^2 \cong T^2/H$  with symplectic orbits.  $\square$

## Symplectic $T$ -model of $(M, \omega)$

Let  $(X, \sigma)$  be a connected symplectic 4-manifold equipped with an effective symplectic action of a 2-torus  $T^2$  for which the  $T^2$ -orbits are symplectic. We give a concise overview of a model of  $(X, \sigma)$ , cf. Section 7.

Consider the quotient map  $\pi : X \rightarrow \widetilde{X/T^2}$ . Choose a base point  $x_0 \in X$ , and let  $p_0 := \pi(x_0)$ . For any homotopy class  $[\gamma] \in \widetilde{X/T^2}$ , i.e., the homotopy class of a loop  $\gamma$  in  $X/T^2$  with base  $p_0$ , denote by

$$\lambda_\gamma : [0, 1] \rightarrow X$$

the unique horizontal lift of  $\gamma$ , with respect to the flat connection  $\Omega$  of symplectic orthogonal complements to the tangent spaces to the  $T$ -orbits, such that  $\gamma(0) = x_0$ , cf. [33, Lemma 3.4.2]. Then the map

$$\Phi : \widetilde{X/T^2} \times T^2 \rightarrow X, \quad ([\gamma], t) \mapsto t \cdot \lambda_\gamma(1)$$

is a smooth covering map and it induces a  $T^2$ -equivariant symplectomorphism

$$\widetilde{X/T^2} \times_{\pi_1^{\text{orb}}(X/T^2)} T^2 \rightarrow X.$$

We refer to see Section 7, and in particular Theorem 7.1 and the remarks below it, for a more thorough description.

## Kähler ingredients

Here we review how a  $T^2$ -invariant complex structure is constructed on a connected symplectic manifold 4-manifold  $(X, \sigma)$  as above.

**Lemma 3.4.** *Consider an effective symplectic action of a 2-torus  $T^2$  on a connected<sup>3</sup> symplectic 4-manifold  $(X, \sigma)$  with a symplectic orbit. Then  $X$  admits a  $T^2$ -invariant Kähler structure. It consists of a  $T^2$ -invariant complex structure, and a Kähler form equal to  $\sigma$ .*

*Proof.* If  $X$  is compact, the lemma is a particular case of Duistermaat-Pelayo [7, Theorem 1.1]. The proof therein is given by constructing the complex structure on the model  $\widetilde{X/T^2} \times_{\pi_1^{\text{orb}}(X/T^2)} T^2$  of  $M$ . In fact, the same proof therein applies even if  $X$  is not compact. We review it for completeness.

The orbifold universal covering  $\widetilde{X/T^2}$  of the orbisurface  $X/T^2$  may be identified with the Riemann sphere, the Euclidean plane, or the hyperbolic plane, on which the orbifold fundamental group  $\pi_1^{\text{orb}}(X/T^2)$  of  $X/T^2$  acts by means of orientation preserving isometries, see Thurston [41, Section 5.5]. Provide  $\widetilde{X/T^2}$  with the standard complex structure.

Let  $\sigma^{T^2}$  be the unique  $T^2$ -invariant symplectic form on  $T^2$  defined by  $\sigma^t$  at the beginning of Section 7: the restriction of  $\sigma$  to any  $T$ -orbit. Equip  $T^2$  with a  $T^2$ -invariant complex structure such that  $\sigma^T$  is equal to a Kähler form.

In this way we obtain a  $T^2$ -invariant complex structure on  $\widetilde{X/T^2} \times_{\pi_1^{\text{orb}}(X/T^2)} T^2$  with symplectic form  $\sigma$  equal to the Kähler form.  $\square$

Suppose that  $a$  is a regular value of  $\mu : M \rightarrow \mathfrak{t}^*$ . Let  $M_a^{\text{reg}}$  be the set of smooth points in  $M_a$ . Then it follows from Lemma 3.4 that  $M_a^{\text{reg}}$  is a smooth manifold that admits a  $T^2$  invariant Kähler structure, which consists of a  $T^2$ -invariant complex structure and a  $T^2$ -invariant Kähler form  $\omega_a$ .

## Duistermaat-Heckman theorem in a complex setting

The following lemma is a key step in our approach.

**Lemma 3.5.** *Consider an effective Hamiltonian action of an  $(n-2)$ -dimensional torus  $T^{n-2}$  with Lie algebra  $\mathfrak{t}$  on a compact, connected,  $2n$ -dimensional symplectic manifold  $(M, \omega)$  with momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . Suppose that there is an effective commuting symplectic  $T^2$ -action on  $M$  which has a symplectic orbit, that  $a$  and  $b$  are two regular values of  $\mu$  which lie in the same connected component of the regular values of the momentum map, and that*

$$M_a := \mu^{-1}(a)/T^{n-2} \text{ and } M_b := \mu^{-1}(b)/T^{n-2}$$

*are symplectic quotients with reduced symplectic structure  $\omega_a$  and  $\omega_b$  respectively. Let  $M_a^{\text{reg}}$  and  $M_b^{\text{reg}}$  be the complement of the sets of orbifold singularities in  $M_a$  and  $M_b$ , respectively.*

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<sup>3</sup>not necessarily compact

Then both  $M_a^{\text{reg}}$  and  $M_b^{\text{reg}}$  are  $T^2$ -invariant Kähler manifolds as in Lemma 3.4, whose Kähler forms are  $\omega_a$  and  $\omega_b$  respectively. Moreover, there is a diffeomorphism  $F : M_a \rightarrow M_b$  such that  $F^*\omega_b$  is a  $(1,1)$ -form on the Kähler manifold  $M_b^{\text{reg}}$ .

*Proof.* By Lemma 3.3, both  $(M_a, \omega_a)$  and  $(M_b, \omega_b)$  are symplectic four orbifolds that admit an effective symplectic  $T^2$  action with symplectic orbits. The smooth parts  $M_a^{\text{reg}}$  and  $M_b^{\text{reg}}$  are symplectic manifolds that admit an effective symplectic action of  $T^2$  with symplectic orbits.

It follows from Lemma 3.4 that  $(M_a^{\text{reg}}, \omega_a)$  and  $(M_b^{\text{reg}}, \omega_b)$  admit a  $T^2$ -invariant complex structure and the corresponding Kähler form is equal to  $\omega_a, \omega_b$ , respectively. We prove that  $F^*\omega_a$  is a  $(1,1)$ -form on  $M_b^{\text{reg}}$  by writing down a local expression of  $F$  using holomorphic coordinates on  $M_a^{\text{reg}}$  and  $M_b^{\text{reg}}$  respectively.

Let  $\Omega_a$  be the flat connection given by the  $\omega_a$ -orthogonal complements to the tangent spaces to the  $T^2$ -orbits on  $M_a^{\text{reg}}$ . Similarly for  $\Omega_b$ . Let  $\Sigma_a := M_a^{\text{reg}}/T^2$ ,  $\Sigma_b := M_b^{\text{reg}}/T^2$ , and let  $\widetilde{\Sigma}_a$  and  $\widetilde{\Sigma}_b$  be the orbifold universal covers of  $\Sigma_a$  and  $\Sigma_b$ , respectively, at some base points.

We have two smooth covering maps

$$\Phi_a : \widetilde{\Sigma}_a \times T^2 \rightarrow M_a^{\text{reg}}$$

and

$$\Phi_b : \widetilde{\Sigma}_b \times T^2 \rightarrow M_b^{\text{reg}}.$$

Suppose  $p_0 = t_0 \lambda_{\gamma_0}(1) \in M_a^{\text{reg}}$  is an arbitrary point. Choose an open neighborhood  $U$  of  $[\gamma_0]$  and  $V$  of  $t_0$  such that the restriction of  $\Phi_a$  to  $U \times V$  is a diffeomorphism. By Theorem 2.5, there exists a  $T^2$ -equivariant diffeomorphism

$$F : M_a^{\text{reg}} \rightarrow M_b^{\text{reg}}.$$

We have that for any  $[\gamma] \in U$  and  $t \in V$ ,

$$F(t\lambda_{\gamma}(1)) = t \cdot F(\lambda_{\gamma}(1)).$$

On the other hand,  $F$  induces a diffeomorphism between two 2-dimensional orbifolds

$$\varphi_{ab} : \Sigma_a \rightarrow \Sigma_b. \tag{4}$$

These orbifolds can be shown to be good orbifolds [33, Lemma 3.4.1], and hence their orbifold universal covers are smooth manifolds. Then we have an induced diffeomorphism between smooth manifolds

$$\widetilde{\varphi}_{ab} : \widetilde{\Sigma}_a \rightarrow \widetilde{\Sigma}_b.$$

Note that  $\widetilde{\varphi}_{ab}(\gamma)$  is a loop based at  $\widetilde{\varphi}_{ab}(p_0)$ . Denote by  $\lambda_{\widetilde{\varphi}_{ab}(\gamma)}$  its horizontal lift with respect to  $\Omega_b$ . Then we have

$$\lambda_{\widetilde{\varphi}_{ab}(\gamma)}(1) = \tau_{\gamma} \cdot F(\lambda_{\gamma}(1)),$$

for some  $\tau_\gamma \in T$ .

Therefore, using covering maps  $\Phi_a$  and  $\Phi_b$ , the diffeomorphism  $F : M_a^{\text{reg}} \rightarrow M_b^{\text{reg}}$  has a local expression

$$([\gamma], t) \mapsto ([\tilde{\varphi}_{ab}(\gamma)], \tau_\gamma t).$$

The proposition follows from the above expression for  $F$ , and how the complex structures and symplectic structures are constructed on  $M_a$  and  $M_b$ , cf. Lemma 3.4.  $\square$

## Walls and Lerman-Guillemin-Sternberg jump formulas

Consider the effective Hamiltonian action of a  $k$ -dimensional torus  $T^k$  on a  $2n$ -dimensional compact symplectic manifold  $(M, \omega)$  with momentum map  $\mu : M \rightarrow \mathfrak{t}^*$ . By Theorem 2.4, the image of the momentum map  $\Delta := \mu(M)$  is a convex polytope. In fact,  $\Delta$  is a union of convex subpolytopes with the property that the interiors of the subpolytopes are disjoint convex open sets and constitute the set of regular values of  $\mu$ . These convex subpolytopes are called the *chambers of  $\Delta$* .

For any circle  $S^1$  of  $T^k$ , and for any connected component  $\mathcal{C}$  of the fixed point submanifold of  $S^1$ , the image of  $\mathcal{C}$  under the momentum map  $\mu$  is called an  $(k-1)$ -dimensional wall, or simply a *wall of  $\Delta$* . Moreover,  $\mu(\mathcal{C})$  is called an *interior wall of  $\Delta$*  if  $\mu(\mathcal{C})$  is not a subset of the boundary of  $\Delta$ .

Now choose a  $T^k$ -invariant inner product on  $\mathfrak{t}^*$  to identify  $\mathfrak{t}^*$  with  $\mathfrak{t}$ . Suppose that  $a$  is a point on a codimension one interior wall  $W$  of  $\Delta$ , and that  $v$  is a normal vector to  $W$  such that the line segment

$$\{a + tv\}$$

is transverse to the wall  $W$ . For  $t$  in a small open interval near 0, write

$$g(t) := \text{DH}_T(a + tv), \tag{5}$$

where  $\text{DH}_T$  is the Duistermaat-Heckman function of  $T$ .

Let  $S^1$  be a circle sitting inside  $T^k$  generated by  $v$ . Let  $H$  be a  $(k-1)$ -dimensional torus  $H$  such that

$$T^k = S^1 \times H.$$

Suppose that  $X$  is a connected component of the fixed point submanifold of the  $S^1$ -action on  $M$  such that  $\mu(X) = W$ . Then  $X$  is invariant under the action of  $H$ . Moreover, the action of  $H$  on  $X$  is Hamiltonian. In fact, let  $\mathfrak{h}$  be the Lie algebra of  $H$ , and let

$$\pi : \mathfrak{t}^* \rightarrow \mathfrak{h}^*$$

be the canonical projection map. Then the composite

$$\mu_H := \pi \circ \mu|_X : X \rightarrow \mathfrak{h}^*$$

is a momentum map for the action of  $H$  on  $X$ .

By assumption,  $a$  is a point in  $W = \mu(X)$  such that  $\pi(a)$  is a regular value of  $\mu_H$ . There are two symplectic quotients associated to  $a$ : one may reduce  $M$  (viewed as an  $H$ -space) at  $\pi(a)$ , and one may reduce  $X$  (viewed as a  $T^k/S^1$ -space) at  $a$ . We denote these symplectic quotients by  $M_a$  and  $X_a$  respectively. It is immediate that  $M_a$  inherits a Hamiltonian  $S^1$ -action and that  $X_a$  is the fixed point submanifold of the  $S^1$ -action on  $M_a$ .

Guillemin-Lerman-Sternberg gave the following formula for computing the jump in the Duistermaat-Heckman function across the wall  $\mu(X)$  of  $\Delta$ .

**Theorem 3.6** (Guillemin-Lerman-Sternberg [12]). *The jump of the function  $g(t)$  in expression (5) when moving across the interior wall  $W$  is given by*

$$g_+(t) - g_-(t) = \sum \text{volume}(X_a) \left( \prod_{i=1}^k \alpha_k^{-1} \right) \frac{t^{k-1}}{(k-1)!},$$

plus an error term of order  $\mathcal{O}(t^k)$ . Here the  $\alpha_k$ 's are the weights of the representation of  $S^1$  on the normal bundle of  $X$ , and the sum is taken over the symplectic quotients of all the connected components  $X$  of  $M^{S^1} \cap \mu^{-1}(\alpha)$  with respect to the  $T^k/S^1$  action at  $a$ .

Building on Theorem 3.6, Graham established the following result, cf. [11, Section 3].

**Proposition 3.7** (Graham [11]). *Suppose that  $\mu : M \rightarrow \mathfrak{t}^*$  is the momentum map of an effective Hamiltonian action of torus  $T$  on a compact, connected symplectic manifold  $(M, \omega)$ . Let  $a$  be a point in a codimension one interior wall of  $\mu(M)$ , and let  $v \in \mathfrak{t}^*$  be a vector such that the line segment  $\{a + tv\}$  is transverse to the wall. For  $t$  in a small open interval near 0, write*

$$g(t) = \text{DH}_T(a + tv),$$

where  $\text{DH}_T$  is the Duistermaat-Heckman function. Then we have that

$$g'_+(0) \leq g'_-(0).$$

## Hodge-Riemann bilinear relations and last step in proof of Theorem 1.4

Now we have all the ingredients to conclude the proof of Theorem 1.4. Our proof depends essentially on the Hodge-Riemann bilinear relation on a Kähler manifold.

Recall that if  $V$  is a vector space and  $A \subset V$  is a convex open subset, and if  $f : A \rightarrow \mathbb{R}$  is a Borel measurable map that is positive on  $A$  almost everywhere, then we say that  $f$  is *log-concave* on  $A$  if and only if  $\log f$  is a concave function on  $A$ . Moreover, if  $f$  is smooth on  $A$ , and if  $a$  is a fixed point in  $A$ , then a simple calculation shows that  $f$  is log-concave on  $A$  if and only if

$$f''(a + tv) \cdot f(a + tv) - (f')^2(a + tv) \leq 0, \text{ for all } v \in A.$$

*Proof of Theorem 1.4.* By Proposition 3.7, to establish the log-concavity of  $\text{DH}_T$  on  $\mu(M)$ , it suffices to show that the restriction of  $\log \text{DH}_T$  to each connected component of the set of regular values of  $\mu$  is concave. Let  $\mathcal{C}$  be such a component, let  $v \in \mathfrak{t}^*$ , and let  $\{a + tv\}$  be a line segment in  $\mathcal{C}$  passing through a point  $a \in \mathcal{C}$ , where the parameter  $t$  lies in some small open interval containing 0. We need to show that

$$g(t) := \text{DH}_T(a + tv)$$

is log-concave, or equivalently, that

$$g''g - (g')^2 \leq 0. \quad (6)$$

Since the point  $a$  is arbitrary, it suffices to show that equation (6) holds at  $t = 0$ . By Theorem 2.5, at  $a + tv \in \mathfrak{t}^*$ , the Duistermaat-Heckman function is

$$\text{DH}_T(a + tv) = \int_{M_a} \frac{1}{2}(\omega_a + tc)^2.$$

Here  $M_a = \mu^{-1}(a)/T$  is the symplectic quotient taken at  $a$  and  $c$  is a closed two-form depending only on  $v \in \mathfrak{t}^*$ . To prove (6), we must show that

$$\int_{M_a} c^2 \int_{M_a} \omega_a^2 \leq 2 \left( \int_{M_a} c \omega_a \right)^2. \quad (7)$$

Consider the primitive decomposition of the two-form  $c$  as

$$c = \gamma + s\omega_a,$$

where  $s$  is a real number, and  $\gamma$  is a primitive two-form. By primitivity, we have that

$$\gamma \wedge \omega_a = 0.$$

A simple calculation shows that (7) for  $\gamma$  implies (7) for  $c$ . But then (7) becomes

$$\int_{M_a} \gamma^2 \leq 0. \quad (8)$$

Note that in a symplectic orbifold the subset of orbifold singularities is of codimension greater than or equal to 2, cf. [3, Prop. III.2.20]. In particular, it is of measure zero with respect to the Liouville measure on the symplectic orbifold. Let  $M_a^{\text{reg}}$  be the complement of the set of orbifold singularities in  $M_a$ . It follows that

$$\int_{M_a} \gamma^2 = \int_{M_a^{\text{reg}}} \gamma^2.$$

Note that  $M_a^{\text{reg}}$  is a connected symplectic four-manifold which admits the effective symplectic action of  $T^2$  with symplectic orbits. By Lemma 3.4,  $M_a^{\text{reg}}$  is a Kähler manifold. Moreover,  $\omega_a$  is the Kähler form on  $M_a^{\text{reg}}$ . In particular, it must be a  $(1, 1)$ -form.

Let  $\langle \cdot, \cdot \rangle$  be the metric on the space of  $(1, 1)$ -forms induced by the Kähler metric, and let  $*$  be the Hodge star operator induced by the Kähler metric. By Lemma 3.5,  $c$  must be a  $(1, 1)$ -form. As a result,  $\gamma$  must be a real primitive  $(1, 1)$ -forms. Applying Weil's identity, cf. [43, Thm. 3.16], we get that

$$\gamma = - * \gamma.$$

Consequently, we have that

$$\begin{aligned} \int_{M_a^{\text{reg}}} \gamma^2 &= - \int_{M_a^{\text{reg}}} \gamma \wedge * \gamma \\ &= - \int_{M_a^{\text{reg}}} \langle \gamma, \gamma \rangle \leq 0. \end{aligned}$$

This completes the proof of Theorem 1.4. □

## Log-concavity conjecture for complexity one Hamiltonian torus actions

We note that our proof of Theorem 1.4 and Theorem 1.3 depend on the classification of symplectic four manifolds with a symplectic  $T^2$  action [33]. However, in the following special case, we have an elementary proof which makes no uses of the classification results therein.

**Theorem 3.8.** *Consider an effective Hamiltonian action of an  $(n - 2)$ -dimensional torus  $T^{n-2}$  on a compact, connected,  $(2n - 2)$ -dimensional symplectic manifold  $(M, \omega)$ . Let  $T^2$  be a 2-dimensional torus equipped with the standard symplectic structure. We consider the following action of  $T^{n-2}$  on the  $2n$ -dimensional product symplectic manifold  $M \times T^2$ ,*

$$g \cdot (m, t) = (g \cdot m, t), \text{ for all } g \in T^{n-2}, m \in M, t \in T^2. \quad (9)$$

*Then the Duistermaat-Heckman function  $\text{DH}_{T^{n-2}}$  of the  $T^{n-2}$  action on  $M \times T^2$  is log-concave.*

*Proof.* Note that there is an obvious commuting  $T^2$  symplectic action on  $M$

$$h \cdot (m, t) = (m, h \cdot t), \text{ for all } h \in T^2, m \in M, t \in T^2, \quad (10)$$

which is effective and which has symplectic orbits. Let  $\mu$  be the momentum map of the  $T^{n-2}$  action on  $M \times T^2$ , and  $\text{DH}_{T^{n-2}}$  the the Duistermaat-Heckman function.

To prove that  $\text{DH}_{T^{n-2}}$  is log-concave, in view of Proposition 3.7, it suffices to show that the restriction of  $\ln \text{DH}_{T^{n-2}}$  to each connected component of the set of regular values of  $\mu$  is concave. Using the same notation as in the proof of Theorem 1.4, we only need to establish the validity of inequality (6). However, a calculation shows that in the current case the function  $g$  must be a degree one polynomial, and so the inequality (6) holds automatically. This completes the proof of Theorem 3.8. □



The following corollary is an immediate consequence of Theorem 3.8.

**Corollary 3.9.** *Suppose that there is an effective Hamiltonian action of an  $(n-1)$ -dimensional torus  $T^{n-1}$  on a compact, connected,  $2n$ -dimensional symplectic manifold  $(M, \omega)$ . Then the Duistermaat-Heckman function  $\text{DH}_T$  of the  $T$ -action is log-concave.*

## 4 Proof of Theorem 1.3

We will use Theorem 1.4 and several other tools (which we explain next), in order to prove Theorem 1.3.

### Logarithmic concavity of torus valued functions

We first extend the notion of log-concavity to functions defined on a  $k$ -dimensional torus  $T^k \cong \mathbb{R}^k / \mathbb{Z}^k$ . Consider the covering map

$$\exp : \mathbb{R}^k \rightarrow T^k, \quad (t_1, \dots, t_k) \mapsto (e^{i2\pi t_1}, \dots, e^{i2\pi t_k}).$$

**Definition 4.1.** We say that a map  $f : T^k \rightarrow (0, \infty)$  is *log-concave at a point*  $p \in T^k$  if there is a point  $x \in \mathbb{R}^k$  with  $\exp(x) = p$  and an open set  $V \subset \mathbb{R}^k$  such that

$$\exp|_V : V \rightarrow \exp(V)$$

is a diffeomorphism and such that the logarithm of

$$\tilde{f} := f \circ \exp$$

is a concave function on  $V$ . We say that  $f$  is *log-concave on*  $T^k$  if it is log-concave at every point of  $T^k$ .

The log-concavity of the function  $f$  does not depend on the choice of  $V$ . Indeed, suppose that there are two points  $x_1, x_2 \in \mathbb{R}^k$  such that

$$\exp x_1 = \exp x_2 = p,$$

and two open sets  $x_1 \in V_1 \subset \mathbb{R}^k$  and  $x_2 \in V_2 \subset \mathbb{R}^k$  such that both

$$\exp|_{V_1} : V_1 \rightarrow \exp(V_1)$$

and

$$\exp|_{V_2} : V_2 \rightarrow \exp(V_2)$$

are diffeomorphisms. Set

$$\tilde{f}_i = f \circ \exp|_{V_i}, \quad i = 1, 2.$$

Then there exists  $(n_1, \dots, n_k) \in \mathbb{Z}^k$  such that

$$\tilde{f}_1(t_1, \dots, t_k) = \tilde{f}_2(t_1 + n_1, \dots, t_k + n_k), \text{ for all } (t_1, \dots, t_k) \in V_1.$$

Now let

$$T^k \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \dots \oplus \mathbb{R}/\mathbb{Z}$$

be a  $k$  dimensional torus, and  $\lambda$  the standard length form on  $S^1 \cong \mathbb{R}/\mathbb{Z}$ . Throughout this section we denote by  $\Theta$  the  $\mathfrak{t}^*$ -valued form

$$\lambda \oplus \lambda \oplus \dots \oplus \lambda \tag{11}$$

on  $T^k$ . Clearly,  $\Theta$  is invariant under the  $T^k$ -action on  $T^k$  by multiplication.

Now consider the action of a  $k$ -dimensional torus  $T^k$  on a  $2n$ -dimensional symplectic manifold  $(M, \omega)$ . Let  $\mathfrak{t}$  be the Lie algebra of  $T^k$ , let  $\mathfrak{t}^*$  be the dual of  $\mathfrak{t}$ , and let  $\Theta$  be the canonical  $T^k$ -invariant  $\mathfrak{t}^*$ -valued one form given as in (11).

**Definition 4.2.** We say that

$$\Phi : M \rightarrow T^k$$

is a *generalized momentum map* if

$$\iota_{X_M} \omega = \langle X, \Phi^* \Theta \rangle,$$

where  $X \in \mathfrak{t}$ , and  $X_M$  is the vector field on  $M$  generated by the infinitesimal action of  $X$  on  $M$ .

## Duistermaat-Heckman densities of Torus valued momentum maps

The following fact is a straightforward generalization of a well-known result due to D. McDuff concerning the existence of circle valued momentum maps.

**Theorem 4.3** (McDuff [28]). *Let a  $k$ -dimensional torus  $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$  act symplectically on the compact symplectic manifold  $(M, \omega)$ . Assume that the symplectic form  $\omega$  represents an integral cohomology class in  $H^2(M, \mathbb{Z})$ . Then:*

- (1) *either the action admits a standard  $\mathbb{R}^k$ -valued momentum map or, if not,*
- (2) *there exists a  $T^k$ -invariant symplectic form  $\omega'$  on  $M$  that admits a  $T^k$ -valued momentum map  $\Phi : M \rightarrow T^k$ .*

When  $\omega$  is integral and  $k = 1$ , the 1-forms  $\iota_X \omega$  is also integral and the map in Theorem 4.3 is defined as follows. Pick a point  $m_0 \in M$ , let  $\gamma_m$  be an arbitrary smooth path connecting  $m_0$  to  $m$  in  $M$ , and define the map  $\Phi : M \rightarrow S^1$  by

$$\Phi(m) := \left[ \int_{\gamma_m} \iota_X \omega \right] \in \mathbb{R}/\mathbb{Z}. \tag{12}$$

One can check that  $\Phi$  is well-defined.

**Remark 4.4.** When  $k = 1$ , a detailed proof of Theorem 4.3 may be found in Pelayo-Ratiu [34]. The argument given in [34] extends to actions of higher dimensional tori, as pointed out therein. As noted in [28], the usual symplectic quotient construction carries through for generalized momentum maps.

Now we will need the following result, which is Theorem 3.5 in [38].

**Theorem 4.5** (Rochon [38]). *Let a compact connected Lie group  $G$  act on a symplectic manifold  $(M, \omega)$  symplectically. Then the  $G$ -action on  $(M, \omega)$  is Hamiltonian if and only if there exists a symplectic form  $\sigma$  on  $M$  such that the  $G$ -action on  $(M, \sigma)$  is Hamiltonian.*

It follows from Theorem 4.5 that in order to show that a symplectic  $S^1$ -action on a symplectic manifold  $(M, \omega)$  is Hamiltonian, it suffices to show it under the assumption that the cohomology class  $[\omega]$  is integral, see [4, Remark 2.1].

We are ready to introduce the following notion.

**Definition 4.6.** Suppose that there is an effective symplectic action of a  $k$ -dimensional torus  $T^k$  on a  $2n$ -dimensional compact symplectic manifold  $(M, \omega)$  with a generalized momentum map  $\Phi : M \rightarrow T^k$ . The *Duistermaat-Heckman measure* is the push-forward of the Liouville measure on  $M$  by  $\Phi$ . Its Density function is the *Duistermaat-Heckman function*.

Analogous to the case of Hamiltonian actions, we have the following result.

**Proposition 4.7.** *Suppose that there is an effective symplectic action of a  $k$ -dimensional torus  $T^k$  on a  $2n$ -dimensional compact symplectic manifold  $(M, \omega)$  with a generalized momentum map  $\Phi : M \rightarrow T^k$ . Let  $(T^k)_{\text{reg}} \subset T^k$  be the set of regular values of  $\Phi$ . Then we have that*

$$\text{DH}_{T^k}(a) = \int_{M_a} \omega_a^{n-k}, \text{ for all } a \in (T^k)_{\text{reg}},$$

where  $M_a := \mu^{-1}(a)/T^k$  is the symplectic quotient taken at  $a$ , and  $\omega_a$  is the reduced symplectic form.

## Symplectic cuttings

We also need the construction of symplectic cutting which was first introduced by Lerman [21]. Next we present a review of this construction.

Let  $\mu : M \rightarrow \mathfrak{t}^*$  be a momentum map for an effective action of a  $k$ -dimensional torus  $T^k$  on a symplectic manifold  $(M, \omega)$  and let  $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{t}$  denote the integral lattice.

Choose  $k$  vectors  $v_j \in \mathfrak{t}_{\mathbb{Z}}$ ,  $1 \leq j \leq k$ . The form

$$\omega - i \sum_j dz_j \wedge d\bar{z}_j$$

is a symplectic form on the manifold  $M \times \mathbb{C}^k$ . The map

$$\nu : M \times \mathbb{C}^k \rightarrow \mathbb{R}^k$$

with  $j$ -th component

$$\nu_j(m, z) = \langle \nu(m), v_j \rangle - |z_j|^2$$

is a momentum map for the  $T^k$ -action on  $M \times \mathbb{C}^k$ .

For any  $b = (b_1, b_2, \dots, b_k) \in \mathbb{R}^k$ , consider a convex rational polyhedral set

$$P = \{x \in \mathfrak{t}^* \mid \langle x, v_i \rangle \geq b_i, \forall 1 \leq i \leq k\}. \quad (13)$$

The *symplectic cut of  $M$  with respect to  $P$*  is the reduction of  $M \times \mathbb{C}^k$  at  $b$ . We denote it by  $M_P$ .

Similarly, for any two regular values  $b = (b_1, \dots, b_k), c = (c_1, \dots, c_k) \in \mathbb{R}^k$  with  $b_i \leq c_i$  for all  $1 \leq i \leq k$ , we define a convex rational polyhedral set

$$Q = \{x \in \mathfrak{t}^* \mid b_i \leq \langle x, v_i \rangle \leq c_i, \forall 1 \leq i \leq k\},$$

and define the *symplectic cut of  $M$  with respect to  $Q$*  to be the reduction of  $M_P$  (as given in the sentence below (13)) at  $c$ .

We note that if  $M$  is in addition compact, then both  $M_P$  and  $M_Q$  are compact.

## Duistermaat-Heckman jump: concluding the proof of Theorem 1.3

*Proof of Theorem 1.3.* We divide the proof into three steps. Suppose that an  $(n-2)$ -dimensional torus  $T^{n-2}$  act on a  $2n$ -dimensional compact connected symplectic manifold  $(M, \omega)$  with fixed points, and that there is a commuting  $T^2$  symplectic action on  $M$  which has a symplectic orbit.

**Step 1.** By Theorem 4.3 and Theorem 4.5, we may assume that there is a generalized momentum map  $\Phi : M \rightarrow T^{n-2}$ . To prove that the action of  $T^{n-2}$  is Hamiltonian, it suffices to prove that for any splitting of  $T^{n-2}$  of the form  $T^{n-2} = S^1 \times H$ , where  $H$  is a  $(n-3)$ -dimensional torus, the action of  $S^1$  is Hamiltonian.

Let  $\pi_1 : T^{n-2} \rightarrow S^1$  and  $\pi_2 : T^{n-2} \rightarrow H$  be the projection maps. Then

$$\phi := \pi_1 \circ \Phi : M \rightarrow S^1 \quad \text{and} \quad \phi_H := \pi_2 \circ \Phi : M \rightarrow H$$

are generalized momentum maps with respect to the  $S^1$ -action and the  $H$ -action on  $M$  respectively. Without loss of generality, we may assume that the generalized momentum map  $\phi$  has no local extreme value, cf. [28, Lemma 2].

Let  $h \in H$  be a regular value of  $\phi_H$ . Denote by  $\text{DH}_{T^{n-2}} : T^{n-2} \rightarrow \mathbb{R}$  the Duistermaat-Heckman function of the  $T^{n-2}$  action on  $M$ . Define

$$f : S^1 \rightarrow \mathbb{R}, \quad f(x) := \text{DH}_{T^{n-2}}(x, h).$$

To prove that the action of  $S^1$  is Hamiltonian, it suffices to show that  $f$  is strictly log-concave. This is because that if  $f$  is strictly log-concave, then the generalized momentum map  $\phi : M \rightarrow S^1$  cannot be surjective, and hence the action of  $S^1$  on  $M$  is Hamiltonian.

We will divide the rest of the proof into two steps, and we will use the following notation. Consider the exponential map  $\exp : \mathbb{R} \rightarrow S^1$ ,  $t \mapsto e^{i2\pi t}$ . By abuse of notation, for any open connected interval  $I = (b, c) \subset \mathbb{R}$  such that  $\exp(I) \not\subseteq S^1$ , we will identify  $I$  with its image in  $S^1$  under the exponential map.

**Step 2.** We first show that if  $I_1 = (b_1, c_1) \not\subseteq S^1$  is an open connected set consisting of regular values of  $\phi$ , then  $f$  is log-concave on  $I_1$ .

Indeed, suppose that for any  $2 \leq i \leq n-2$ ,  $I_i = (b_i, c_i) \not\subseteq S^1$  is an open connected set such that

$$\mathcal{O} = I_2 \times \cdots \times I_{n-2} \ni h$$

consists of regular values of  $\phi_H$ . Then  $I_1 \times \mathcal{O} \not\subseteq T^{n-2}$  is a connected open set that consists of regular values of  $\Phi : M \rightarrow T^{n-2}$ . Moreover, the action of  $T^{n-2}$  on

$$W = \Phi^{-1}(I \times \mathcal{O})$$

is Hamiltonian. Set

$$Q = \{(x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2} \mid b_i \leq x_i \leq c_i, \forall 1 \leq i \leq n-2\}.$$

The symplectic cutting  $W_Q$  of  $W$  with respect to  $Q$  is a compact Hamiltonian manifold from  $W$ , and, on the interior of  $Q$ , the Duistermaat-Heckman function of  $W_Q$  agrees with that of  $W$ . Consequently, the log-concavity of the function  $f$  follows from Theorem 1.4.

**Step 3.** We show that if  $I_1 = (b_1, c_1) \not\subseteq S^1$  is a connected open set which contains a unique critical value  $a$  of  $\phi$ , then  $f$  is strictly log-concave on  $I_1$ . We will use the same notation  $W$  and  $W_Q$  as in the Step 2. By definition, to show that  $f$  is strictly log-concave on  $I_1$ , it suffices to show that

$$\tilde{f} = f \circ \exp : \mathbb{R} \rightarrow \mathbb{R}$$

is strictly log-concave on the pre-image of  $I_1$  in  $\mathbb{R}$  under the exponential map  $\exp$ . Again, by abuse of notation, we would not distinguish  $I_1$  and its pre-image in  $\mathbb{R}$ , and we will use the same notation  $a$  to denote the pre-image of  $a \in I_1 \not\subseteq S^1$  in  $\exp^{-1}(I_1)$ .

To see the existence of such interval  $I_1$ , we need the assumption that the fixed point sets of  $T^{n-2}$  is non-empty. Since the fixed point submanifold of the action of  $T^{n-2}$  on  $M$  is a subset of  $M^{S^1}$ , it follows that  $M^{S^1}$  is non-empty. Let  $X$  be a connected component of  $M^{S^1}$ . Choose  $a \in S^1$  to be the image of  $X$  under the generalized momentum map  $\phi$ . Then a sufficiently small connected open subset that contains  $a$  will contain no other critical values of  $\phi$ .

Note that  $X$  is a symplectic submanifold of  $M$  which is invariant under the action of  $H$ , and that there are two symplectic manifolds associated to the point  $(a, h) \in T^k = S^1 \times H$ :

the symplectic quotient of  $M$  (viewed as an  $H$ -space) at  $h$ , and the symplectic quotient of  $X$  (viewed as  $T^k/S^1$ -space) at  $(a, h)$ . We will denote these two quotient spaces by  $M_h$  and  $X_h$  respectively. An immediate calculation shows that the dimension of  $M_h$  is six.

Since the action of  $S^1$  commutes with that of  $H$ , by Lemma 3.1 and Remark 3.2, the action of  $S^1$  preserves the level set of the generalized momentum map  $\phi_H$ . Consequently, there is an induced action of  $S^1$  on  $M_h$ . Since  $\phi : M \rightarrow S^1$  has a constant value on any  $H$  orbit in  $M$ , it descends to a generalized momentum map

$$\tilde{\phi} : M_h \rightarrow S^1.$$

It is easy to see that  $X_h$  consists of the critical points of the generalized momentum map  $\tilde{\phi} : M_h \rightarrow S^1$ . Moreover, since  $\phi$  has no local extreme values in  $M$ , it follows that  $\tilde{\phi}$  has no local extreme value in  $M_h$ . Thus for dimensional reasons,  $X_h$  can only be of codimension four or six.

Now observe that the commuting symplectic action of  $T^2$  on  $M$  descends to a symplectic action on  $M_h$  which commutes with the action of  $S^1$  on  $M^{S^1}$ . It follows that  $X_h$  is invariant under the induced action of  $T^2$  on  $M_h$  and so must contain a  $T^2$  orbit. As a result,  $X_h$  must have codimension four.

Applying Theorem 3.6 to the compact Hamiltonian manifold  $W_Q$ , we get that

$$\tilde{f}_+(a+t) - \tilde{f}_-(a-t) = \sum_{X_h} \frac{\text{vol}(X_h)}{(d-1)! \prod_j \alpha_j} (t-a)^{d-1} + \mathcal{O}((t-a)^d).$$

In the above equation,  $X_h$  runs over the collection of all connected component of  $M^{S^1}$  that sits inside  $\Phi^{-1}((a, h))$ , the  $\alpha_j$ 's are the weights of the representation of  $S^1$  on the normal bundle of  $X_h$ , and  $d$  is half of the real dimension of  $X_h$ .

By our previous work,  $d = 2$  for any  $X_h$ ; moreover, the two non-zero weights of  $X_h$  must have opposite signs. So the jump in the derivative is strictly negative, i.e.,

$$\tilde{f}'_+(a) - \tilde{f}'_-(a) < 0.$$

This completes the proof of Theorem 1.3. □

## 5 Examples and consequences of main results

### Applications of Theorems 1.3 and 1.4

Our proof of Theorem 1.3 depends on Lemma 3.4. We note that the proof of Lemma 3.4 builds on the classification result of symplectic  $T^2$  action on symplectic four manifolds, cf. [33] and [7]. However, we have a simpler proof in the following special case, which was the goal of the article [20] (see Section 1 therein).

**Theorem 5.1** (Kim [20]). *Let  $(M, \omega)$  be a compact, connected, symplectic  $2n$ -dimensional manifold. Then every effective symplectic action of an  $(n - 1)$ -dimensional torus  $T^{n-1}$  on  $(M, \omega)$  with non-empty fixed point set is Hamiltonian.*

*Proof.* We prove the equivalent statement that if  $(N, \omega)$  is  $(2n - 2)$ -dimensional compact, connected symplectic manifold, then every symplectic action of an  $(n - 2)$ -dimensional torus  $T^{n-2}$  on  $(N, \omega)$  with non-empty fixed point set is Hamiltonian.

Indeed, suppose that there is a symplectic action of an  $(n - 2)$ -dimensional torus on a  $(2n - 2)$ -dimensional compact connected symplectic manifold  $N$ . By Theorem 4.3 and Theorem 4.5, without loss of generality we assume that there is a generalized momentum map

$$\phi_N : N \rightarrow T^{n-2}.$$

Consider the action of  $T^{n-2}$  on the product symplectic manifold  $Z := N \times T^2$  as given in equation (9), where  $T^2$  is equipped with the standard symplectic structure. Define

$$\Phi : N \times T^2 \rightarrow T^{n-2}, \quad (x, t) \mapsto \phi_N(x), \text{ for all } x \in N, \text{ for all } t \in T^2.$$

Then  $\Phi$  is a generalized momentum map for the symplectic action of  $T^{n-2}$  on  $Z$ . Moreover, there is an effective commuting symplectic action of  $T^2$  on  $M$  with symplectic orbits given as in equation (10).

To show that the action of  $T^{n-2}$  on  $N$  is Hamiltonian, it suffices to show that the action of  $T^{n-2}$  on  $Z$  is Hamiltonian. Using the same notation as in the proof of Theorem 1.3, we explain that we can modify the argument there to prove that the action of  $T^{n-2}$  on  $Z$  is Hamiltonian without using Lemma 3.4. We first note that the step 2 in the proof of Theorem 1.3 does not use Lemma 3.4, and that the same argument there applies to the present situation, which is a special case of what is considered in Theorem 1.3.

Lemma 3.4 is used in the step 1 in the proof of Theorem 1.3 to prove that  $f$  is log-concave on  $I_1$ . However, in the present situation, we can use Theorem 3.8 instead to establish the log-concavity of the function  $f$  on  $I_1$ . Since the proof of Theorem 3.8 does not use Lemma 3.4, Theorem 5.1 can be established without using Lemma 3.4 as well.  $\square$

The following result [28, Proposition 2] is a special case of Theorem 5.1.

**Corollary 5.2** (McDuff [28]). *If  $(M, \omega)$  is 4-dimensional compact, connected symplectic manifold, then every effective symplectic  $S^1$ -action on  $(M, \omega)$  with non-empty fixed point set is Hamiltonian.*

Using [37] we obtain the following result on measures. A measure defined on the measurable subsets of  $\mathbb{R}^n$  is *log-concave* if for every pair  $A, B$  of convex subsets of  $\mathbb{R}^n$  and for every  $0 < t < 1$  such that  $tA + (1 - t)B$  is measurable we have

$$P(tA + (1 - t)B) \geq (P(A))^t (P(B))^{1-t}$$

where the sign  $+$  denotes Minkowski addition of sets.



**Theorem 5.3** (Prékopa [37]). *If the density function of a measure  $P$  defined on the measurable subsets of  $\mathbb{R}^n$  is almost everywhere positive and is a log-concave function, then the measure  $P$  itself is log-concave.*

These notions extend immediately to our setting where  $\mathbb{R}^n$  is replaced by the dual Lie algebra  $\mathfrak{t}^*$ , equipped with the push-forward of the Liouville measure on  $M$ , and we have, as a consequence of Theorem 1.4, the following.

**Theorem 5.4.** *Let  $T$  be an  $(n - 2)$ -torus which acts effectively on a compact connected symplectic  $2n$ -manifold  $(M, \omega)$  in a Hamiltonian fashion. Suppose that there is a commuting symplectic action of a 2-torus  $S$  on  $M$  whose orbits are symplectic. Then the Duistermaat-Heckman measure is log-concave.*

## Examples

Next we give some more examples to which our theorems apply.

**Example 5.5.** The Kodaira-Thurston manifold is the symplectic manifold

$$\text{KT} := (\mathbb{R}^2 \times T^2 / \mathbb{Z}^2).$$

This is the case in Theorem 1.3 which is “trivial” since  $n - 2 = 0$ , and hence  $T^{n-2} = \{e\}$  (in fact this manifold is non-Kähler, and as such it does not admit any Hamiltonian torus action, of any non-trivial dimension).

Nevertheless KT serves to illustrate a straightforward case which admits the symplectic transversal symmetry, and it may be used to construct lots of examples satisfying the assumptions of the theorem, eg.  $\text{KT} \times S^2$ .

**Example 5.6.** This example is a generalization of Example 5.5. Theorem 1.3 covers an infinite class of symplectic manifolds with Hamiltonian  $S^1$ -actions, for instance

$$M := ((\mathbb{CP}^1)^2 \times \mathbb{T}^2) / \mathbb{Z}_2, \tag{14}$$

where  $S^1$  acts Hamiltonianly on the left factor  $(\mathbb{CP}^1)^2$ , and  $\mathbb{T}^2$  acts symplectically with symplectic orbits on the right factor; the quotient is taken with respect to the natural diagonal action of  $\mathbb{Z}_2$  (by the antipodal action on a circle of  $\mathbb{T}^2$ , and by rotation by 180 degrees about the vertical axes of the spheres  $S^2$ ), and the action on  $(\mathbb{CP}^1)^2 \times \mathbb{T}^2$  descend to an action on  $M$ . Any product symplectic form upstairs also descends to the quotient.

**Example 5.7.** The  $2n$ -dimensional symplectic manifolds with  $\mathbb{T}^n$ -actions in Theorem 1.3 are a subclass of those classified in [33], which, given that  $\mathbb{T}^2$  is 2-dimensional, are of the form  $M_Z$  described in (15).

Let  $Z$  be any good  $(2n - 2)$ -dimensional orbifold, and let  $\tilde{\Sigma}$  be its orbifold universal cover. Equip  $Z$  with any orbifold symplectic form, and  $\tilde{\Sigma}$  with the induced symplectic form. Equip  $\mathbb{T}^2$  with any area form. Equip  $\tilde{Z} \times \mathbb{T}^2$  with the product symplectic form, and the symplectic

$\mathbb{T}^2$ -action with symplectic orbits by translations on the right most factor. This symplectic  $\mathbb{T}^2$ -action descends to a symplectic  $\mathbb{T}^2$ -action with symplectic orbits on

$$M_Z := \tilde{Z} \times_{\pi_1^{\text{orb}}(Z)} \mathbb{T}^2 \quad (15)$$

considered with the induced product symplectic form.

An a priori intractable question is: *give an explicit list of manifolds  $M_Z$  which admit a Hamiltonian  $S^1$ -action.* On the other hand, examples of such  $M_Z$ , and hence fitting in the statement of Theorem 1.3, may be constructed in all dimensions, eg. the manifold in (14).

Theorem 1.3 says that any commuting symplectic  $\mathbb{T}^{n-2}$  action on (15) which has fixed points, must be Hamiltonian.

## 6 Final Remarks

### The theorems of Frankel and McDuff

The relation between conditions (i) and (ii) in Section 1 can be traced back to a result proved by T. Frankel in 1959, which says that if  $(M, \omega)$  is  $2n$ -dimensional compact Kähler manifold, then every symplectic  $S^1$ -action on  $(M, \omega)$  with fixed points is Hamiltonian [8]. In 1986 D. McDuff showed that if  $(M, \omega)$  is 4-dimensional symplectic manifold, then every symplectic  $S^1$ -action on  $(M, \omega)$  with fixed points is Hamiltonian [28, Proposition 2].

In the same paper, McDuff gave an example of a symplectic  $S^1$ -action on a compact connected symplectic six manifold  $(M, \omega)$  which has non-empty fixed point set but is not Hamiltonian. However, the fixed point set in her example is a submanifold, and to this day there are no known examples of non-Hamiltonian symplectic  $S^1$ -actions on 6-manifolds (or manifolds of any dimension  $2n \geq 6$ ) which have a discrete fixed point set. This has given rise to what some authors have called the McDuff-Frankel question: is there such an example, or, on the contrary, is every such action Hamiltonian?

A large number of papers have been written on the subject, but to this day the answer is, as far as we know, unknown. The answer is known to be yes for semifree actions, see Tolman-Weitsman [40, Theorem 1].

Feldman [10, Theorem 1] characterized the obstruction for a symplectic circle action on a compact manifold to be Hamiltonian and deduced the McDuff and Tolman-Weitsman theorems by applying his criterion. He showed that the Todd genus of a manifold admitting a symplectic circle action with isolated fixed points is equal either to 0, in which case the action is non-Hamiltonian, or to 1, in which case the action is Hamiltonian. In addition, any symplectic circle action on a compact manifold with positive Todd genus is Hamiltonian. For additional results regarding aspherical symplectic manifold see [19, Section 8] and [27].

Ono proved that if  $(M, \omega)$  is a compact connected symplectic  $S^1$ -manifold which is a Lefschetz manifold<sup>4</sup>, then if the  $S^1$ -action has fixed points, then it is Hamiltonian. Y. Lin

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<sup>4</sup>Let  $(M, \omega)$  be a symplectic  $2n$ -manifold and, for each  $k = 0, \dots, n$ , consider the linear map

$$H^{n-k}(M, \mathbb{R}) \ni [\alpha] \mapsto [\alpha \wedge \omega^k] \in H^{n+k}(M, \mathbb{R}).$$

proved recently [26, Theorem 4.5] that there exist infinitely many topologically inequivalent 6-dimensional compact Lefschetz manifolds with Hamiltonian  $S^1$ -actions with fixed points, and which are not homotopy equivalent to any compact Kähler manifold. Hence within the category of compact connected symplectic manifolds we have an inclusion

$$\{S^1\text{-Kähler manifolds with fixed points}\} \subsetneq \{S^1\text{-Lefschetz manifolds with fixed points}\}$$

and hence Ono's Theorem covers a class of manifolds which is strictly larger than that covered in Frankel's theorem.

More recently, Cho, Hwang, and Suh [5] studied symplectic circle actions on compact connected 6-manifolds  $(M, \omega)$  with non-isolated fixed points. Assume that the cohomology class  $[\omega]$  is integral and that there exists a circle valued momentum map  $\mu : M \rightarrow S^1$ . These authors observed that if the Duistermaat-Heckman function of the circle valued momentum map  $F_\mu : S^1 \rightarrow \mathbb{R}$  is log-concave on  $S^1$ , then it must be a constant. However, if the fixed point set is non-empty, the Duistermaat-Heckman function can not be a constant since there will be a non-zero jump in its value moving across a critical level set. In particular, combining this observation with a result in Lin [22] concerning the log-concavity of the Duistermaat-Heckman function, they proved that if the regular symplectic quotient taken with respect to the circle valued momentum map satisfying  $b_+ = 1$ , then the symplectic circle action must be Hamiltonian if its fixed point set is non-empty.

Finally, in the context of orbifolds, an analogue of Tolman and Weitsman's result was obtained by Godinho [17].

## Results on higher dimensional groups

For higher dimensional Lie groups, some results are known, see for instance Giacobbe [9, Theorem 3.13], Duistermaat-Pelayo [7, Corollary 3.9] and Ginzburg [15, Proposition 4.2]. A Hodge theoretic approach for non-compact manifolds was introduced by Pelayo-Ratiu recently [34, Theorem 1].

## Minimal fixed point set of symplectic actions

The McDuff-Frankel question is related to the question of what is the minimal number of fixed points of a symplectic circle action that has non-empty fixed point set. If the action is Hamiltonian, this is easily shown to be  $n + 1$  using Morse theory for the momentum map of the action.

Motivated by the work of Tolman and Weitsman on semifree actions, in which they use equivariant cohomological methods to approach this question, Pelayo-Tolman used a technique [36, Lemma 8] involving Vandermonde determinants. Using this technique they showed that the number of fixed points of a symplectic circle action with non-empty fixed

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If the map for  $k = n - 1$  is an isomorphism, then  $(M, \omega)$  is called a *Lefschetz manifold*. If all these maps for  $k = 0, \dots, n$  are isomorphisms, then  $(M, \omega)$  is called a *strong Lefschetz manifold*.

point set on a compact symplectic  $2n$ -dimensional manifold is at least  $n+1$  (see [36, Theorem 1]), provided the Chern class map is somewhere injective (the *Chern class map*  $M^{S^1} \rightarrow \mathbb{Z}$  assigns to a fixed point the sum of the weights of the action at that point). Recently, several authors have continued this equivariant cohomological approach, proving a number of interesting results, see for instance Li [23, Theorem 1.9], Li-Liu [24, 25] and the references therein.

## 7 Appendix: actions with symplectic orbits

This section is a review of [33, Sections 2, 3], with a new remark at the end. The remark is that many of the results therein extends to the case of symplectic torus actions with symplectic orbits on non-necessarily compact symplectic manifolds  $(X, \omega)$ . We needed this remark to prove Theorem 1.4.

Let  $(X, \sigma)$  be a connected symplectic manifold equipped with an effective symplectic action of a torus  $T$  for which there is at least one  $T$ -orbit which is a  $\dim T$ -dimensional symplectic submanifold of  $(X, \sigma)$  (this implies that all are). *We do not assume that  $X$  is necessarily compact.* Then there exists a unique non-degenerate antisymmetric bilinear form  $\sigma^{\mathfrak{t}}: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$  on the Lie algebra  $\mathfrak{t}$  of  $T$  such that  $\sigma_x(u_X(x), v_X(x)) = \sigma^{\mathfrak{t}}(u, v)$ , for every  $u, v \in \mathfrak{t}$ , and every  $x \in X$ .

One can check that the stabilizer subgroup  $T_x$  of the  $T$ -action at every point  $x \in X$  is a finite group.

### Orbit space $X/T$

As usual,  $X/T$  denotes the orbit space of the  $T$ -action. Let  $\pi: X \rightarrow X/T$  the canonical projection. The space  $X/T$  is provided with the maximal topology for which  $\pi$  is continuous; this topology is Hausdorff. Because  $X$  is connected,  $X/T$  is connected.

### Orbifold atlas on $X/T$

Let  $k := \dim X - \dim T$ . By the tube theorem of Koszul (see eg. [14, Theorem B24]), for each  $x \in X$  there exists a  $T$ -invariant open neighborhood  $U_x$  of the  $T$ -orbit  $T \cdot x$  and a  $T$ -equivariant diffeomorphism  $\Phi_x$  from  $U_x$  onto the associated bundle  $T \times_{T_x} D_x$ , where  $D_x$  is an open disk centered at the origin in  $\mathbb{R}^k \cong \mathbb{C}^{k/2}$  and  $T_x$  acts by linear transformations on  $D_x$ .

The action of  $T$  on  $T \times_{T_x} D_x$  is induced by the action of  $T$  by translations on the left factor of  $T \times D_x$ . Because  $\Phi_x$  is a  $T$ -equivariant diffeomorphism, it induces a homeomorphism  $\hat{\Phi}_x$

on the quotient  $\widehat{\Phi}_x: D_x/T_x \rightarrow \pi(U_x)$ , and there is a commutative diagram of the form

$$\begin{array}{ccccc} T \times D_x & \xrightarrow{\pi_x} & T \times_{T_x} D_x & \xrightarrow{\Phi_x} & U_x \\ i_x \uparrow & & \downarrow p_x & & \downarrow \pi|_{U_x} \\ D_x & \xrightarrow{\pi'_x} & D_x/T_x & \xrightarrow{\widehat{\Phi}_x} & \pi(U_x) \end{array}$$

where  $\pi_x, \pi'_x, p_x$  are the canonical projection maps, and  $i_x(d) := (e, d)$  is the inclusion map. Let  $\phi_x := \widehat{\Phi}_x \circ \pi'_x$ . The collection of charts

$$\widehat{\mathcal{A}} := \{(\pi(U_x), D_x, \phi_x, T_x)\}_{x \in X}$$

is an orbifold atlas for  $X/T$ . We call  $\mathcal{A}$  the class of atlases equivalent to the orbifold atlas  $\widehat{\mathcal{A}}$ . We denote the orbifold  $X/T$  endowed with the class  $\mathcal{A}$  by  $X/T$ , and the class  $\mathcal{A}$  is assumed.

## Flat connection

The collection  $\Omega = \{\Omega_x\}_{x \in X}$  of subspaces  $\Omega_x \subset T_x X$ , where  $\Omega_x$  is the  $\sigma_x$ -orthogonal complement to  $T_x(T \cdot x)$  in  $T_x X$ , for every  $x \in X$ , is a smooth distribution on  $X$ . The projection mapping  $\pi: X \rightarrow X/T$  is a smooth principal  $T$ -orbifold for which  $\Omega$  is a  $T$ -invariant flat connection. Let  $\mathcal{I}_x$  be the maximal integral manifold of the distribution  $\Omega$ .

The inclusion  $i_x: \mathcal{I}_x \rightarrow X$  is an injective immersion between smooth manifolds and the composite  $\pi \circ i_x: \mathcal{I}_x \rightarrow X/T$  is an orbifold covering map. Moreover, there exists a unique 2-form  $\nu$  on  $X/T$  such that  $\pi^* \nu|_{\Omega_x} = \sigma|_{\Omega_x}$  for every  $x \in X$ . The form  $\nu$  is symplectic, and so the pair  $(X/T, \nu)$  is a connected symplectic orbifold.

## Model for $X/T$

We define the space that we call the  *$T$ -equivariant symplectic model*  $(X_{\text{model}, p_0}, \sigma_{\text{model}})$  of  $(X, \sigma)$  based at a regular point  $p_0 \in X/T$  as follows.

- i) The space  $X_{\text{model}, p_0}$  is the associated bundle

$$X_{\text{model}, p_0} := \widetilde{X/T} \times_{\pi_1^{\text{orb}}(X/T, p_0)} T,$$

where the space  $\widetilde{X/T}$  denotes the orbifold universal cover of the orbifold  $X/T$  based at a regular point  $p_0 \in X/T$ , and the orbifold fundamental group  $\pi_1^{\text{orb}}(X/T, p_0)$  acts on the Cartesian product  $\widetilde{X/T} \times T$  by the diagonal action  $x(y, t) = (x \star y^{-1}, \mu(x) \cdot t)$ , where  $\star: \pi_1^{\text{orb}}(X/T, p_0) \times \widetilde{X/T} \rightarrow \widetilde{X/T}$  denotes the natural action of  $\pi_1^{\text{orb}}(X/T, p_0)$  on  $\widetilde{X/T}$ , and  $\mu: \pi_1^{\text{orb}}(X/T, p_0) \rightarrow T$  denotes the monodromy homomorphism of  $\Omega$ .

- ii) The symplectic form  $\sigma_{\text{model}}$  is induced on the quotient by the product symplectic form on the Cartesian product  $\widetilde{X/T} \times T$ . The symplectic form on  $\widetilde{X/T}$  is defined as the pullback by the orbifold universal covering map  $\widetilde{X/T} \rightarrow X/T$  of  $\nu$ . The symplectic form on the torus  $T$  is the unique  $T$ -invariant symplectic form  $\sigma^T$  determined by  $\sigma^t$ .
- iii) The action of  $T$  on the space  $X_{\text{model}, p_0}$  is the action of  $T$  by translations which descends from the action of  $T$  by translations on the right factor of the product  $\widetilde{X/T} \times T$ .

In this definition we are implicitly using that the universal cover  $\widetilde{X/T}$  is a smooth manifold and the orbit space  $X/T$  is a good orbifold, which is proven by analogy with [33].

## Model of $(X, \sigma)$ with $T$ -action

In [33], the following theorem was shown (see Theorem 3.4.3).

**Theorem 7.1** (Pelayo [33]). *Let  $(X, \sigma)$  be a compact connected symplectic manifold equipped with an effective symplectic action of a torus  $T$ , for which at least one, and hence every  $T$ -orbit is a  $\dim T$ -dimensional symplectic submanifold of  $(X, \sigma)$ . Then  $(X, \sigma)$  is  $T$ -equivariantly symplectomorphic to its  $T$ -equivariant symplectic model based at any regular point  $p_0 \in X/T$ .*

The purpose of this appendix was to point out that Theorem 7.1 with the word “compact” removed from the statement holds. We refer to [33] for a proof in the case that  $X$  is compact, which works verbatim in the case that  $X$  is not compact.

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